Exact Description of Decoherence in Optical Cavities

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Abstract

The exact reduced dynamics for the independent oscillator model in the RWA approximation at zero and finite temperatures is derived. It is shown that the information about the interaction and the environment is encapsulated into three time dependent coefficients of the master equation, one of which vanishes in the zero temperature case. In currently used optical cavities all the information about the field dynamics is contained into *two* (or three) experimentally accesible and physically meaningful real functions of time. From the phenomenological point of view it suffices then to carefully measure two (*three*) adequate observables in order to map the evolution of any initial condition, as shown with several examples: (generalized) coherent states, Fock states, Schrödinger cat states, and squeezed states.

1 Introduction

Measuring the time development of the entanglement process of a system coupled to its environment is a most remarkable achievement and a challenging goal. The reason for this is that the entanglement process is a unique and typical quantum feature. Several attemps, both on the theoretical as well as on the experimental side have been recently made[1, 2]. In the particular case of high Q optical cavities a direct measure of the decoherence process has been given and suggestions of experiments with essentially the same set up have been made on how to directly measure the Wigner function of the initially correlated field produced in the cavity[3, 4]. We show that all necessary information to construct the time development of any Wigner function (or any system's density operator) can be obtained by a precise measure of three quantities as function of time: the average photon number and two orthogonal field quadratures.

Recently the exact master equation for quantum Brownian motion in a general environment has been derived using both path integral techniques[5] and the tracing of the evolution equation for the Wigner function[6, 7]. We closely follow the later approach to derive the exact master equation for the oscillator independent model in the RWA at zero temperature and with a factorized initial condition. The hamiltonian of the model is

$$H = \hbar\omega(a^\dagger a + 1/2) + \hbar\sum_k \omega_k(a_k^\dagger a_k + 1/2) + \hbar\sum_k c_k(a^\dagger a_k + a_k^\dagger a). \tag{1}$$

Here we present the solution of the initial value problem, which have been solved in the Heisenberg picture in Refs. [8, 9]. The solution allows for an easy visualization in contrast to the model without the RWA

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approximation. Besides its intrinsic interest as an exactly soluble model, the hamiltonian (1) may be useful in treating leaking Bose-Einstein condensates [10], in materials with modified dispersion relations[11], or in any case of non-ohmic strength function, where the Born-Markov approximation is not adequate[10].

We assume that at $t=0$ the total density operator is given by

$$\rho(0) = \rho_S(0) \otimes \prod_k \frac{e^{-\beta\hbar\omega_k a_k^\dagger a}}{\mathrm{Tr} e^{-\beta\hbar\omega_k a_k^\dagger a}} \overset{\beta\to\infty}{\longrightarrow} \rho_S(0) \otimes \prod_k |0_k\rangle\langle 0_k|, \tag{2}$$

where the subscript S for system refers to the main oscillator. The bath, i.e. the set of oscillators labelled by k, is initially in thermal equilibrium at inverse temperature β. At zero temperature, the tensor product of the vacuum of the main oscillator with the vacuum of the set of oscillators is the ground state of (1).

2 The Exact Master Equation

It is well known that for quadratic hamiltonians the Wigner function satisfies the classical Liouville equation. To obtain the classical hamiltonian corresponding to (1), one uses the correspondence rule $a_\mu^{(\dagger)} \to \alpha_\mu^{(*)}$, where

$$\alpha_\mu^{(*)} = \sqrt{\frac{m_\mu\omega_\mu}{2\hbar}} q_\mu + (-)\frac{i}{\sqrt{2\hbar m_\mu\omega_\mu}} p_\mu, \tag{3}$$

and discards the zero energy contributions. We use $\mu = 0,1,2,..,$ $k = 1,2,..$ and $a_0 \equiv a$, $\omega_0 = \omega$. In general greek subindices denote non negative integers while latin sub indices denote positive integers. Using these conventions, the classical Liouville equation

$$\frac{\partial W(\alpha_\mu,\alpha_\mu^*,t)}{\partial t} = \frac{1}{i\hbar}\sum_\mu \frac{\partial H}{\partial\alpha_\mu}\frac{\partial W}{\partial\alpha_\mu^*} - \frac{1}{i\hbar}\sum_\mu \frac{\partial H}{\partial\alpha_\mu^*}\frac{\partial W}{\partial\alpha_\mu}, \tag{4}$$

where $W(\alpha_\mu,\alpha_\mu^*,t)$ is the Wigner function in the quantum case and the probability density function in the classical case, can be written as

$$\begin{aligned}\frac{\partial W(\alpha_\mu,\alpha_\mu^*,t)}{\partial t} \quad = \quad & -i\omega\alpha^*\frac{\partial W}{\partial\alpha^*} + i\omega\alpha\frac{\partial W}{\partial\alpha} - i\sum_k \omega_k\alpha_k^*\frac{\partial W}{\partial\alpha_k^*} + i\sum_k \omega_k\alpha_k\frac{\partial W}{\partial\alpha_k} \\ & -i\sum_k c_k\alpha_k^*\frac{\partial W}{\partial\alpha^*} + i\sum_k c_k\alpha_k\frac{\partial W}{\partial\alpha} - i\sum_k c_k\alpha^*\frac{\partial W}{\partial\alpha_k^*} + i\sum_k c_k\alpha\frac{\partial W}{\partial\alpha_k}.\end{aligned} \tag{5}$$

The initial condition Eq. (2) in the language of Wigner functions is

$$W(\alpha_\mu,\alpha_\mu^*,t=0) = W^0(\alpha_\mu,\alpha_\mu^*) = W_S^0(\alpha,\alpha^*)W_B^0(\alpha_k,\alpha_k^*) = W_S^0(\alpha,\alpha^*)\prod_k N_k \mathrm{e}^{-2\tanh(\hbar\omega_k\beta)\alpha_k\alpha_k^*}, \tag{6}$$

where the N_k are normalization constants. Integrating Eq.(5) over the bath variables we get

$$\frac{\partial\widetilde{W}(\alpha,\alpha^*,t)}{\partial t} = -i\omega\alpha^*\frac{\partial\widetilde{W}}{\partial\alpha^*} + i\omega\alpha\frac{\partial\widetilde{W}}{\partial\alpha} - i\frac{\partial G^*}{\partial\alpha^*} + i\frac{\partial G}{\partial\alpha}, \tag{7}$$

with

$$\widetilde{W}(\alpha,\alpha^*,t) = \int\left(\prod_k d\alpha_k d\alpha_k^*\right) W(\alpha_\mu,\alpha_\mu^*,t), \tag{8}$$

and

$$G(\alpha,\alpha^*,t) = \int\left(\prod_k d\alpha_k d\alpha_k^*\right)\sum_k c_k\alpha_k W(\alpha_\mu,\alpha_\mu^*). \tag{9}$$

As in [7] it is easy to show that $G(\alpha,\alpha^*,t)$ can be written in terms of $\widetilde{W}$. We notice that for quadratic hamiltonians $W(\alpha_\mu,\alpha_\mu^*,t)=W^0(\alpha_\mu(-t),\alpha_\mu^*(-t))$, where $\alpha_\mu(t)$ is the solution of the classical equations of motion. If we define $\vec{\alpha}(t)=(\alpha_0(t),\alpha_1(t),\ldots)$ and denote its transpose by $\vec{\alpha}^T(t)$, we have

$$\vec{\alpha}^T(t)=U^\dagger\Delta(t)U\vec{\alpha}^T(0), \qquad \vec{\alpha}^*(t)=(\vec{\alpha}(t))^*, \tag{10}$$

where U and Δ are unitary, and Δ is diagonal. Taking the Fourier transform of G, and changing variables from $\{\alpha_\mu(-t),\alpha_\mu^*(-t)\}$ to $\{\alpha_\mu(0),\alpha_\mu^*(0)\}$, with unit Jacobian, we obtain

$$\begin{aligned} G(\kappa,\kappa') = & \int\prod_\mu d\alpha_\mu(0)d\alpha_\mu^*(0)\mathrm{e}^{i\kappa\sum_\nu p_\nu(t)\alpha_\nu(0)}\mathrm{e}^{i\kappa'\sum_\nu p_\nu^*(t)\alpha_\nu^*(0)} \\ & \times\sum_\nu q_\nu(t)\alpha_\nu(0)W_S^0(\alpha^{(*)}(0))W_B^0(\alpha_k^{(*)}(0)), \end{aligned} \tag{11}$$

with $\{p_\nu,p_\nu^*,q_\nu\}$ time dependent parameters. From Eq.(11) it is easy to see that the multiplication by $\alpha_0=\alpha$ is equivalent to a derivation with respect to k, plus terms corresponding to multiplication by α_k, up to time dependent coefficients. These last terms, as can be seen from (6), correspond to derivations w.r.t. α_k^*, which in turn, are equivalent to multiplication by k', as shows a simple integration by parts. Taking the inverse Fourier transform we obtain a multiplication by α and a derivation w.r.t. α^*. Thus, observing that the Fourier transform of $\widetilde{W}(\alpha,\alpha^*)$, $\widetilde{W}(\kappa,\kappa')$ is given by

$$\begin{aligned} \widetilde{W}(\kappa,\kappa') = & \int\prod_\mu d\alpha_\mu(0)d\alpha_\mu^*(0)\mathrm{e}^{i\kappa\sum_\nu p_\nu(t)\alpha_\nu(0)}\mathrm{e}^{i\kappa'\sum_\nu p_\nu^*(t)\alpha_\nu^*(0)} \\ & \times W_S^0(\alpha^{(*)}(0))W_B^0(\alpha_k^{(*)}(0)), \end{aligned}$$

we obtain

$$\begin{aligned} i\frac{\partial G}{\partial\alpha}-i\frac{\partial G^*}{\partial\alpha^*} &= iY\frac{\partial}{\partial\alpha}\left(\alpha\widetilde{W}\right)-iY^*\frac{\partial}{\partial\alpha^*}\left(\alpha^*\widetilde{W}\right) \\ &\quad +(iZ-iZ^*)\frac{\partial^2\widetilde{W}}{\partial\alpha\partial\alpha^*}, \end{aligned} \tag{12}$$

with time dependent functions Y,Z. Therefore the Wigner equation can be written as

$$\begin{aligned} \frac{\partial\widetilde{W}(\alpha,\alpha^*,t)}{\partial t} &= -i(\omega+\delta)\left(\alpha^*\frac{\partial\widetilde{W}}{\partial\alpha^*}-\alpha\frac{\partial\widetilde{W}}{\partial\alpha}\right)+2\lambda\widetilde{W} \\ &\quad +\lambda\left(\alpha^*\frac{\partial\widetilde{W}}{\partial\alpha^*}+\alpha\frac{\partial\widetilde{W}}{\partial\alpha}\right)+\lambda'\frac{\partial^2\widetilde{W}}{\partial\alpha\partial\alpha^*}, \end{aligned} \tag{13}$$

where we have set $iY=\lambda+i\delta$ and $iZ-iZ^*=\lambda'$. All of λ,δ and λ' are real functions. By comparing the system of equations found from both (5) and (13) we get

$$(\lambda+i\delta)\langle\alpha\rangle = i\sum_k c_k\langle\alpha_k\rangle, \tag{14}$$

$$(\lambda+i\delta)\langle\alpha^2\rangle = i\sum_k c_k\langle\alpha\alpha_k\rangle, \tag{15}$$

$$-2\lambda'\langle\alpha\alpha^*\rangle = i\sum_k c_k\langle\alpha\alpha_k^*\rangle-i\sum_k c_k\langle\alpha^*\alpha_k\rangle. \tag{16}$$

We know that the solution of the Heisenberg equations can be written as follows

$$a(t) = \eta(t)a(0)+\sum_k\gamma_k(t)a_k(0), \tag{17}$$

$$a_k(t) = \eta_k(t)a(t)+\sum_l\gamma_{kl}a_l(0). \tag{18}$$

Using the above solution of the Heisenberg equations, and the fact that all of the first and second (symmetric) moments involving bath operators are zero, with the exception of $\langle\{a_k^\dagger, a_k\}\rangle = 2n_k + 1 = 2\coth(\hbar\omega_k\beta/2)$, we obtain

$$\begin{aligned} \lambda(t) + i\delta(t) &= i\sum_k c_k\eta_k(t) && (19)\\ \lambda'(t) &= \sum_{kl} c_k(n_k(\beta) + \frac{1}{2})(i\gamma_l\gamma_{kl}^* - i\gamma_l^*\gamma_{kl}). && (20) \end{aligned}$$

For reasons that will be clear soon, we write the diffusion coefficient $\lambda'(t)$ as $\lambda(t)+\epsilon(t,\beta)$, with $\lim_{\beta\to\infty}\epsilon(t,\beta) = 0$, as shown in section 5. At zero temperature, since the tensor product of vacua is the ground state of (1), the corresponding (reduced) Wigner function $W_S(\alpha,\alpha^*)$ should be a stationary solution of the Wigner equation. When this condition is applied to Eq.(13), we obtain $\lambda = \lambda'$. If they were not equal it would imply the non existence of an exact master equation, as seems to be claimed in Ref. [10]. However, this is not the case, as we show next.

It is not hard to show that the operator equation for the system's reduced density operator equivalent to the Wigner equation (13) is

$$\frac{d\rho}{dt} = \frac{1}{i\hbar}\left[\hbar(\omega+\delta)a^\dagger a, \rho\right] + (\lambda+\epsilon)(2a\bullet a^\dagger - a^\dagger a\bullet - \bullet a^\dagger a)\rho + \epsilon(2a^\dagger\bullet a - aa^\dagger\bullet - \bullet aa^\dagger)\rho = \mathcal{L}(t)\rho(t), \quad (21)$$

where the usual dot superoperator convention has been used. The usual Born-Markov RWA master equation is of this form with constant coefficients[12]. Some results can be obtained at once from (21): premultiplying by a and taking the trace we get

$$\frac{d}{dt}\langle a\rangle = \frac{d\alpha}{dt} = (-i(\omega+\delta) - \lambda)\alpha, \quad (22)$$

which can be immediately solved to give

$$\alpha(t) = \exp(-i\Omega(t) - \Lambda(t))\alpha(0), \quad (23)$$

with

$$\Omega(t) = \int_0^t d\tau(\omega+\delta)(\tau), \qquad \Lambda(t) = \int_0^t d\tau\lambda(\tau). \quad (24)$$

Note that this result is *independent* of ϵ, i.e., it does not depend on the temperature. Premultiplying (21) by $a^\dagger a$, and taking the trace we get the following differential equation

$$\frac{d}{dt}\langle a^\dagger a\rangle(t) = -2\lambda\langle a^\dagger a\rangle(t) + 2\epsilon \quad (25)$$

with the solution

$$\langle a^\dagger a\rangle(t) = \exp(-2\Lambda(t))\langle a^\dagger a\rangle(0) + \mathcal{N}(t) = \langle a^\dagger a\rangle(t;\beta\to\infty) + 2\exp(-2\Lambda(t))\int_0^t d\tau\epsilon(\tau)\exp(2\Lambda(\tau)), \quad (26)$$

where it is evident that $\mathcal{N}(t)$ vanishes in the zero temperature limit. Contrary to the exact equations found in [5, 6, 7] We thus have the following interpretations for the real functions that appear in the master equation: $\delta(t)$ is the instantaneous frequency shift, $\lambda(t)$ is the instantaneous energy rate of change at zero temperature and $\epsilon(t)$ is the instantaneous energy rate of change at finite temperature but with the system in the vacuum state. Moreover $\mathcal{N}(t)$, which is related to both $\epsilon(t)$ and $\delta(t)$ is the mean number of excitations when the initial state was the ground state.

3 The evolution superoperator and some initial states

We can use Lie algebraic methods[13] to find the evolution superoperator $\mathcal{U}$. Indeed, we can verify that the superoperators $\mathcal{M} = a^\dagger a \bullet$, $\mathcal{P} = \bullet a^\dagger a$, $\mathcal{J} = a \bullet a^\dagger$ and $\mathcal{R} = a^\dagger \bullet a$ form an algebra,

$$[\mathcal{M}, \mathcal{P}] = 0, \quad [\mathcal{M}, \mathcal{J}] = -\mathcal{J} = [\mathcal{P}, \mathcal{J}], \quad [\mathcal{M}, \mathcal{R}] = -\mathcal{R} = [\mathcal{P}, \mathcal{R}]. \tag{27}$$

Thus, we can assume that $\mathcal{U}(t) = v \mathrm{e}^{w\mathcal{R}} \mathrm{e}^{x\mathcal{M}} \mathrm{e}^{y\mathcal{P}} \mathrm{e}^{z\mathcal{J}}$. Deriving this expression, using the formula $\exp(xA) B \exp(-xA) = B + x[A, B] + x^2[B, [B, A]]/2! + ...$ and the commutation relations (27), comparing coefficients in the equation $d\mathcal{U}/dt = \mathcal{L}(t)\mathcal{U}(t)$, and solving the resulting differential equations we obtain

$$\begin{aligned} v(t) &= \frac{1}{1+\mathcal{N}(t)}, \quad w(t) = \frac{\mathcal{N}(t)}{1+\mathcal{N}(t)}, \quad x(t) = -i\Omega(t) - \Lambda(t) - \frac{1}{2}\ln(1+\mathcal{N}(t)) = y^*(t), \\ & z(t) = 1 - \frac{\exp(-2\Lambda(t))}{1+\mathcal{N}(t)}. \end{aligned} \tag{28}$$

Let us suppose that ρ satisfies the equation $d\rho/dt = \mathcal{L}(X_i\bullet, \bullet X_i; t)\rho(t)$, where the X_i are operators (notice that $\mathcal{L}$ is a general linear superoperator), and that ρ can be written as $U\rho' U^{-1}$. Then, ρ' satisfy the equation $d\rho'/dt = \mathcal{L}'\rho'(t)$. If, moreover, we choose $U = \exp(\sigma a^\dagger - \sigma^* a) = D(\sigma)$, the displacement operator, and $\mathcal{L}$ is that of the RWA, then

$$\mathcal{L}'(t) = \mathcal{L}(t) + \left((i\dot{\Omega} + \lambda)\sigma + \frac{d\sigma}{dt}\right)(a^\dagger \bullet - \bullet\, a^\dagger) - \left((-i\dot{\Omega} + \lambda)\sigma^* + \frac{d\sigma^*}{dt}\right)(a \bullet - \bullet\, a). \tag{29}$$

It is easy to see that if

$$\sigma(t) = \sigma(0)\exp(-i\Omega(t) - \Lambda(t)), \qquad \sigma^*(t) = (\sigma(t))^*, \tag{30}$$

then both ρ and ρ' satisfy the same equation. That is, we have shown that $D(\sigma(t))\rho(t)D^\dagger(\sigma(t))$ satisfies Eq. (21) whenever $\rho(t)$ does the same. We remark that this result does *not* depend on the temperature of the bath.

We now turn to the evaluation of the density matrix evolved with the superoperator found above. We chose initial states relevant from the point of view of quantum optics. As a first initial state we choose the system's ground state, $\rho(0) = \mid 0\rangle\langle 0 \mid$. Since

$$\mathrm{e}^{x\mathcal{M}} \mid 0\rangle\langle 0 \mid \; = \; \mathrm{e}^{y\mathcal{P}} \mid 0\rangle\langle 0 \mid = \mathrm{e}^{z\mathcal{J}} \mid 0\rangle\langle 0 \mid = \mid 0\rangle\langle 0 \mid, \quad \text{and} \tag{31}$$

$$\mathrm{e}^{x\mathcal{R}} \mid 0\rangle\langle 0 \mid \; = \; \sum_0^\infty \frac{x^n}{n!}(a^\dagger)^n \mid 0\rangle\langle 0 \mid a^n = \sum_0^\infty x^n \mid n\rangle\langle n \mid, \tag{32}$$

we have

$$\rho(t) = \mathcal{U}(t) \mid 0\rangle\langle 0 \mid = \sum_0^\infty \frac{1}{1+\mathcal{N}}\,(1+1/\mathcal{N})^n \mid n\rangle\langle n \mid = \sum_0^\infty P_n(t) \mid n\rangle\langle n \mid. \tag{33}$$

The above formula displays the so called decomposition in natural orbits[15] where the quantities P_n can be directly interpreted as probabilities. We can write the evolved density matrix in the alternative form $\rho(t) = \exp((1+1/\mathcal{N})a^\dagger a)/(1+\mathcal{N})$, which is the form of an instantaneous thermal density matrix, with $\mathcal{N}(t) = < a^\dagger a > (t)$. Had we chosen an initial thermal state, with mean number of excitations $\bar{n}(0)$, the density matrix would have remained a thermal state, but now $M(t) = \bar{n}(0)\exp(-2\Lambda(t)) + \mathcal{N}(t)$. If we use the instantaneous oscillator frequency $\omega' = \omega + \delta$, it is possible to define an instantaneous temperature through the relation $T(t) = \hbar(\omega+\delta)/(k_B \ln(1+1/\mathcal{N}))$, with k_B the Boltzmann constant. Moreover, we have obtained a physical interpretation for the quantity $\mathcal{N}(t)$: it is the mean number of excitations of the main oscillator at time t when it was initially prepared in its ground state.

To calculate the density matrix for an initial Fock state it is better to write the evolution superoperator in the form $\mathcal{U}(t) = v\exp(w\mathcal{R})\exp(z'\mathcal{J})\exp(x\mathcal{M})\exp(y\mathcal{P})$, where w, x, y are given in (28), and $z'(t) = (1+\mathcal{N})\exp(2\Lambda)$. We use

$$\begin{aligned} e^{z\mathcal{J}}e^{x\mathcal{M}}e^{y\mathcal{P}} \mid m\rangle\langle m \mid &= \sum_{k=0}^{m}(e^{x+y})^{m-k}(ze^{x+y})^k\frac{m!}{(m-k)!k!} \mid m-k\rangle\langle m-k \mid \\ &= \sum_{k=0}^{m}\frac{m!}{(m-k)!k!}(e^{x+y})^k(ze^{x+y})^{m-k} \mid k\rangle\langle k \mid, \quad \textit{and} \qquad (34) \\ e^{u\mathcal{R}} \mid m\rangle\langle m \mid &= \sum_{k=0}^{\infty}\frac{m!u^k}{(m-k)!k!} \mid k\rangle\langle k \mid, \qquad (35) \end{aligned}$$

to see that the density matrix at time t is given by $\rho(t) = \sum_{k=0}^{\infty} P_{m,s}(t) \mid s\rangle\langle s \mid$, with

$$P_{m,s}(t) = \frac{e^{-2m\Lambda}}{(1+\mathcal{N})^{m+1}}\frac{m!}{s!}\sum_{k=0}^{\min(m,s)}\frac{([1+\mathcal{N}]e^{2\Lambda}-1)^k}{(m-k)!(s-k)!}\left(1+\frac{1}{\mathcal{N}}\right)^{s-k}. \qquad (36)$$

Since the former density matrix have been expressed in terms of natural orbits, the quantities $P_{m,s}(t)$ are readily interpreted as probabilities. The transformation property discussed above allows us to write the evolution of an initial generalized coherent state $\mid \sigma m\rangle = D(\sigma) \mid m\rangle$, where D is the displacement operator and $\mid n\rangle$ the n-th number state. We have $\mathcal{U}(t) \mid \sigma_0 m\rangle\langle\sigma_0 m \mid = \sum_{k=0}^{\infty} P_{m,s}(t) \mid \sigma(t)s\rangle\langle\sigma(t)s \mid$, with $P_{m,s}(t)$ given by (36) and $\sigma(t)$ by (30).

One interesting point to be investigated is if there exists an asymptotic density operator. Provided that our environment is such that $\lim_{t\uparrow\infty}\Lambda(t) \to \infty$ and $\lim_{t\uparrow\infty}\mathcal{N}(t) = n_\infty$, the asymptotic evolution superoperator can be written as

$$\lim_{t\uparrow\infty}\mathcal{U}(t) = \frac{1}{1+n_\infty}\exp(\frac{n_\infty}{1+n_\infty}\mathcal{R})(\mid 0\rangle\langle 0 \mid \bullet)\exp(\mathcal{J})(\bullet \mid 0\rangle\langle 0 \mid), \qquad (37)$$

which applied to a generic normalized initial density $\rho(0)$ gives

$$\begin{aligned} \rho_\infty &= \frac{1}{1+n_\infty}\exp(\mathcal{J})\rho(0)\langle 0 \mid \exp(\frac{n_\infty}{1+n_\infty}\mathcal{R}) \mid 0\rangle\langle 0 \mid \\ &= \frac{1}{1+n_\infty}(\mathrm{Tr}\rho(0))\exp[(1+\frac{1}{n_\infty})a^\dagger a] \mid 0\rangle\langle 0 \mid = \frac{1}{1+n_\infty}\exp[(1+\frac{1}{n_\infty})a^\dagger a] \mid 0\rangle\langle 0 \mid. \qquad (38) \end{aligned}$$

Thus, whenever the established conditions are met, the density operator approaches asymptotically to a thermal state with a mean number of excitations equal to that of the environment. The existence of a unique asymptotic density can not be taken for granted: in the model of decoherence without damping studied in references [14] even when the coefficient of decoherence grows indefinitely with time, the asymptotic state depends on the initial state.

The normal order characteristic functional$C^{(n)}(\xi,\xi^\dagger,t)$ given by [12]

$$C^{(n)}(\xi,\xi^\dagger,t) = \mathrm{Tr}e^{i\xi a^\dagger}e^{i\xi^* a}\rho(t) = \mathrm{Tr}e^{i\xi a^\dagger}e^{i\xi^* a}\mathcal{U}(t)\rho(0) = \mathrm{Tr}\mathcal{U}^\dagger(t)e^{i\xi a^\dagger}e^{i\xi^* a}\rho(0), \qquad (39)$$

where $\mathcal{U}(t)$ is the evolution superoperator and $\mathcal{U}^\dagger(t)$ its adjoint, is the generating functional of the normally ordered moments. After a somewhat lengthy but straightforward calculation we obtain

$$C^{(n)}(\xi,\xi^\dagger,t) = e^{-2\mathcal{N}\xi xi^\dagger}\mathrm{Tr}e^{i\xi\exp(-\Lambda+i\Omega)a^\dagger}e^{i\xi^*\exp(-\Lambda-i\Omega)a}\rho(0). \qquad (40)$$

If we calculate $\langle a\rangle(t)$ and $\langle a\dagger a\rangle(t)$ using

$$\langle a\rangle(t) = \frac{\partial}{\partial\xi^*}C^{(n)}(\xi,\xi^\dagger,t)\mid_{\xi=0=\xi^\dagger}, \quad \langle a\dagger a\rangle(t) = \frac{\partial^2}{\partial\xi\partial\xi^*}C^{(n)}(\xi,\xi^\dagger,t)\mid_{\xi=0=\xi^\dagger}, \qquad (41)$$

we arrive at the same results as before. These can be compared to those obtained in reference [8].

4 The Zero Temperature Limit

The zero temperature limit has its own special interest, both as an approximation at low temperatures, and as the relevant case for leaking Bose-Einstein condensates. In this case the evolution superoperator can be expressed as $\mathcal{U}(t) = \mathrm{e}^{\tilde{x}\mathcal{M}}\mathrm{e}^{\tilde{y}\mathcal{P}}\mathrm{e}^{\tilde{z}\mathcal{J}}$ with

$$\tilde{x}(t) = -i\Omega(t) - \Lambda(t) = \tilde{y}^*(t), \quad \tilde{z}(t) = 1 - \exp(-2\Lambda(t)). \tag{42}$$

Since the vacuum is solution of the master equation, the transformation property discussed above indicates that

$$D(\sigma(t))|0\rangle\langle 0|D^\dagger(\sigma(t)) = |\sigma(t)\rangle\langle\sigma(t)| = \mathcal{U}(t)|\sigma_0\rangle\langle\sigma_0| \tag{43}$$

also solves the master equation. That means that initial coherent states evolve preserving their coherence, not matter what the details of the environment. A measurement of the norm and the phase of an initial coherent state is enough to determine the functions Ω and Λ, and hence, in principle, to determine the evolution of any other initial state.

The evolution of an initial Fock state is also easily calculated.

$$\mathcal{U}(t)|m\rangle\langle m| = \sum_{k=0}^{m} p_{k,m}(t)|k\rangle\langle k| = p_{k,m}(t) = \frac{m!}{k!(m-k)!}(\mathrm{e}^{-2\Lambda(t)})^k(1-\mathrm{e}^{-2\Lambda(t)})^{m-k}|k\rangle\langle k|. \tag{44}$$

From this solution we can generate another solution: if we have an initial generalized coherent state, $|m;\sigma_0\rangle = D(\sigma_0)|m\rangle$, it evolves into a mixture of generalized coherent states, as given by

$$\mathcal{U}(t)|m;\sigma_0\rangle\langle m;\sigma_0| = \sum_{k=0}^{m} p_{k,m}(t)|k;\sigma(t)\rangle\langle k;\sigma(t)|. \tag{45}$$

Applying the evolution operator $\mathcal{U}(t)$ to an initial density matrix element $|\sigma_0\rangle\langle\sigma'_0|$, one obtains

$$\mathcal{U}(t)|\sigma_0\rangle\langle\sigma'_0| = \frac{\langle\sigma'_0|\sigma_0\rangle}{\langle\sigma'(t)|\sigma(t)\rangle}|\sigma(t)\rangle\langle\sigma'(t)|. \tag{46}$$

Notice that an heuristic argumentation also leads to (46). Indeed, since a coherent state remains coherent, we expect $\mathcal{U}(t)|\sigma_0\rangle\langle\sigma'_0| = N(t)|\sigma(t)\rangle\langle\sigma'(t)|$. As long as the exact dynamical equation for ρ preserves the trace, $\frac{d}{dt}\mathrm{Tr}\rho_S(t) = \mathrm{Tr}(\frac{d\rho_S(t)}{dt}) = \mathrm{Tr}(\mathcal{L}(t)\rho_S(t)) = 0$, the normalization factor $N(t)$ cannot be other than that of Eq. (46). The evolution of an initial even ($\rho_{\sigma_0 e}$) or odd cat ($\rho_{\sigma_0 o}$) state can be calculated from (46) and (43). Indeed,

$$\rho_{\sigma_0 e(o)} = N_{e(o)}(\sigma_0)(|\sigma_0\rangle, |-\sigma_0\rangle)\begin{pmatrix} 1 & (-)1 \\ (-)1 & 1 \end{pmatrix}\begin{pmatrix} \langle\sigma_0| \\ \langle-\sigma_0| \end{pmatrix}, \tag{47}$$

where $N_{e(o)}(\sigma_0) = (1 + (-)\langle-\sigma_0|\sigma_0\rangle)^{-1}/2$ is a normalization factor, evolves as follows

$$\mathcal{U}\rho_{\sigma_0 e(o)} = N_{e(o)}(\sigma_0)(|\sigma_t\rangle, |-\sigma_t\rangle)\begin{pmatrix} 1 & \frac{(-)\langle-\sigma_0|\sigma_0\rangle}{\langle-\sigma(t)|\sigma(t)\rangle} \\ \frac{(-)\langle-\sigma_0|\sigma_0\rangle}{\langle-\sigma(t)|\sigma(t)\rangle} & 1 \end{pmatrix}\begin{pmatrix} |\sigma_t\rangle \\ |-\sigma_t\rangle \end{pmatrix}. \tag{48}$$

We can rewrite Eq. (48) in a more convenient way, in terms of natural orbitals, as

$$\mathcal{U}(t)\rho_{\sigma_0 e(o)} = p^{e(o)}_{e(o)}(t)\rho_{\sigma(t)e(o)} + p^{o(e)}_{e(o)}(t)\rho_{\sigma(t)o(e)}, \tag{49}$$

with

$$p^{e(o)}_{e(o)}(t) = \frac{1}{2}\frac{1+(-)\langle-\sigma(t)|\sigma(t)\rangle}{1+(-)\langle-\sigma_0|\sigma_0\rangle}\left(1 + \frac{\langle-\sigma_0|\sigma_0\rangle}{\langle-\sigma(t)|\sigma(t)\rangle}\right) \tag{50}$$

$$p^{o(e)}_{e(o)}(t) = \frac{1}{2}\frac{1-(-)\langle-\sigma(t)|\sigma(t)\rangle}{1+(-)\langle-\sigma_0|\sigma_0\rangle}\left(1 - \frac{\langle-\sigma_0|\sigma_0\rangle}{\langle-\sigma(t)|\sigma(t)\rangle}\right). \tag{51}$$

Observe that an initial cat state evolves as a mixture of even and odd cat states. Eq. (50) gives the probability of the cat state of the same parity as the initial state, and Eq. (51) the probability of the cat state of the other parity.

The evolution of an initial squeezed state is also easily computed. We notice that the density operator for the squeezed vacuum $\rho(\zeta)$, can be written as

$$\rho(\zeta) = \lim_{\gamma\to\infty} \rho(\zeta,\gamma) = \lim_{\gamma\to\infty} (1-e^{-\gamma}) S(\zeta) e^{-\gamma a^\dagger a} S^\dagger(\zeta), \tag{52}$$

where $S(\zeta) = \exp((\zeta(a^\dagger)^2 - \zeta^\dagger a^2)/4)$. Setting $\zeta = \xi \exp i\phi$, the following expressions for the second moments of $\rho(\zeta,\gamma)$ are found

$$\langle (a^\dagger)^2\rangle = \frac{e^{-i\phi}}{2}\sinh(\xi), \qquad \langle a^2\rangle = \frac{e^{i\phi}}{2}\sinh(\xi), \qquad \langle\{a,a^\dagger\}\rangle = \cosh(\xi). \tag{53}$$

The system of equations for the second moments,

$$\frac{d\langle a^2\rangle}{dt} = -2\frac{d}{dt}(i\Omega+\Lambda)\langle a^2\rangle, \tag{54}$$

$$\frac{d\langle (a^\dagger)^2\rangle}{dt} = 2\frac{d}{dt}(i\Omega-\Lambda)\langle (a^\dagger)^2\rangle, \tag{55}$$

$$\frac{d\langle\{a,a^\dagger\}\rangle}{dt} = -2\frac{d\Lambda}{dt}\langle\{a,a^\dagger\}\rangle + \frac{d\Lambda}{dt}, \tag{56}$$

is integrated to give

$$\langle (a^2)^{(\dagger)}\rangle(t) = \exp(-(-)2i\Omega - 2\Lambda)\langle (a^2)^{(\dagger)}\rangle(0), \tag{57}$$

$$\langle\{a,a^\dagger\}\rangle(t) = \exp(-2\Lambda)\langle\{a,a^\dagger\}\rangle(0) + (1-\exp(-2\Lambda))/2. \tag{58}$$

From these relationships we find $\mathcal{U}(t)\rho(\zeta) = \rho(\zeta(t),\gamma(t))$, with

$$\zeta(t) = \xi(t)\exp(i\phi(t)) \qquad \phi(t) = \phi_0 - 2\Omega \tag{59}$$

$$\xi(t) = \mathrm{ArcTanh}(\frac{\sinh(\xi_0)}{\cosh(\xi_0)+\exp(2\Lambda)-1}) \tag{60}$$

$$\gamma(t) = \mathrm{ArcCoth}(\sqrt{e^{-4\Lambda} + 2\cosh(\xi_0)e^{-2\Lambda}(1-e^{-2\Lambda}) + (1-e^{-2\Lambda})^2}). \tag{61}$$

Now the evolution of a general initial state $\rho(\sigma,\zeta) = D(\sigma)\rho(\zeta)D^\dagger(\sigma)$ is

$$\mathcal{U}(t)\rho(\sigma,\zeta) = \rho(\sigma(t),\zeta(t),\gamma(t)) = D(\sigma(t))\rho(\zeta(t),\gamma(t))D^\dagger(\sigma(t)), \quad \text{or} \tag{62}$$

$$\mathcal{U}(t)\left(|\sigma_0,\zeta_0,n=0\rangle\langle\sigma_0,\zeta_0,n=0|\right) = D(\sigma(t))S(\zeta(t))(1-e^{-\gamma(t)})e^{-\gamma(t)a^\dagger a}S^\dagger(\zeta(t))D^\dagger(\sigma(t)), \tag{63}$$

where $\sigma(t)$, $\zeta(t)$ and $\gamma(t)$ are those of Eqs. (30) and (59–61), and

$$|\sigma_0,\zeta_0,n\rangle = D(\sigma(t))S(\zeta(t))|n\rangle. \tag{64}$$

The corresponding expansion in terms of natural orbits is

$$\begin{aligned}\mathcal{U}(t)\left(|\sigma_0,\zeta_0,0\rangle\langle\sigma_0,\zeta_0,0|\right) &= \sum_{n=0}^{\infty} p_n(t)|\sigma(t),\zeta(t),n\rangle\langle\sigma(t),\zeta(t),n| \\ &= \sum_{n=0}^{\infty} e^{-n\gamma(t)}(1-e^{-\gamma(t)})|\sigma(t),\zeta(t),n\rangle\langle\sigma(t),\zeta(t),n|. \end{aligned} \tag{65}$$

5 Determination of the Master Equation Coefficients

Let's return to Eq.(10). If we call w_μ the exact eigenfrequencies of (1), we can write (remenber that greek indices can assume the value 0, while latin indices do not)

$$a_\nu(t) = \sum_{\mu\sigma} U^*_{\mu\nu} U_{\mu\sigma} \mathrm{e}^{-iw_\mu t} a_\sigma(0) = \sum_\sigma Z_{\nu\sigma} a_\sigma(0). \tag{66}$$

For the sake of convenience we write

$$\eta = Z_{00}, \quad \gamma_k = Z_{0k}, \quad \Delta_k = Z_{k0}, \quad \Gamma_{kl} = Z_{kl}. \tag{67}$$

Using (66) in Eqs. (17) and (18), we obtain the following expressions for η_k and γ_{kl},

$$\eta_k = \frac{\Delta_k}{\eta} \qquad \gamma_{kl} = \Gamma_{kl} - \frac{\Delta_k \gamma_l}{\eta}. \tag{68}$$

From (19) and (20) we obtain

$$\lambda = -\sum_k c_k \mathrm{Im}(\eta_k), \qquad \lambda' = -\sum_k c_k (2n_k(\beta)+1)\mathrm{Im}(\beta_k) = -\sum_k c_k(2n_k(\beta)+1)\mathrm{Im}(\sum_l \gamma_l \gamma^*_{kl}). \tag{69}$$

Since U is unitary we have

$$\begin{aligned}\sum_l \gamma_l \Gamma^*_{kl} &= \sum_{\lambda\mu\nu} U^*_{\mu 0} U_{\mu\lambda} U_{\nu k} U^*_{\nu\lambda} \mathrm{e}^{-i(w_\mu - w_\nu)t} - \sum_{\mu\nu} U^*_{\mu 0} U_{\mu 0} U_{\nu k} U^*_{\nu 0} \mathrm{e}^{-i(w_\mu - w_\nu)t} \\ &= \sum_{\mu\nu} U^*_{\mu 0} U_{\nu k} \mathrm{e}^{-i(w_\mu - w_\nu)t} \delta_{\mu\nu} - \eta \Delta^*_k = \sum_\mu U^*_{\mu 0} U_{\mu k} - \eta\Delta^*_k = -\eta \Delta^*_k,\end{aligned} \tag{70}$$

and

$$\sum_l \gamma_l \gamma^*_l = \sum_{\lambda\mu\nu} U^*_{\mu 0} U_{\mu\lambda} U_{\nu 0} U^*_{\nu\lambda} \mathrm{e}^{-i(w_\mu - w_\nu)t} - \sum_{\mu\nu} U^*_{\mu 0} U_{\mu 0} U_{\nu 0} U^*_{\nu 0} \mathrm{e}^{-i(w_\mu - w_\nu)t} = 1 - \eta\eta^*. \tag{71}$$

Using (70) and (71) we get

$$\beta_k = \sum_l \gamma_l \gamma^*_{kl} = \sum_l \gamma_l \Gamma^*_{kl} - \frac{\Delta^*_k}{\eta^*} \sum_l \gamma_l \gamma^*_l - \eta \Delta^*_k - \frac{\Delta^*_k}{\eta^*}(1 - \eta\eta^*) = -\eta^*_k. \tag{72}$$

Thus we finally notice that $\lambda'(t)$ can be expressed as a sum

$$\lambda'(t) = -\sum_k c_k(2n_k(\beta)+1)\mathrm{Im}(\eta_k) = \lambda(t) - 2\sum_k c_k n_k(\beta) \mathrm{Im}(\eta_k) = \lambda(t) + \epsilon(t;\beta). \tag{73}$$

Observe that $\lim_{\beta\to\infty} \epsilon(t;\beta) = 0$. This proves that indeed we have the equality $\lambda(t) = \lambda'(t)$ at zero temeperature, as can be seen by looking at their definitions as given in Eq. (69).

We have seen that δ, λ and ϵ (or Ω, Λ and $\mathcal{N}$) are all we need to characterize the effect of the bath on the main oscillator. The Heisenberg equations of motion,

$$\begin{aligned}\dot{a} &= -i\omega_0 a - i\sum_k c_k a_k, \\ \dot{a_k} &= -i\omega_k a_k - i c_k a,\end{aligned} \tag{74, 75}$$

can be solved in a number of ways. For example, an implicit method gives

$$a(t) = a(0) e^{-i\Omega(t) - \Lambda(t)} - i \sum_k c_k a_k(0) \int_0^t d\tau e^{-i\omega_k(t-\tau)} e^{-i\Omega(\tau)-\Lambda(\tau)}. \tag{76}$$

where $\eta(t)$ satisfies the integrodifferential equation

$$\dot{\eta} + i\omega\eta + \int_0^t d\tau \sum_k c_k^2 \mathrm{e}^{i\omega_k(t-\tau)}\eta(\tau) = 0, \tag{77}$$

subject to the initial condition $eta_0 = 1$. Using the $\eta_k(t)$ implicitly defined above in eq (19), and taking into account the equation satidfied by η, we obtain $\lambda + i\delta = -i\omega + d(\ln \eta)/dt$, or $\eta(t) = \exp(-\Lambda(t) - i\Omega(t)$. We also find

$$\epsilon(t) = \frac{e^{-2\Lambda(t)}}{2}\frac{d}{dt}\left(e^{2\Lambda(t)}\sum_k c_k^2 \mid \int_0^t d\tau e^{-i\omega_k(t-\tau)} e^{-i\Omega(\tau)-\Lambda(\tau)} \mid^2 n_k(\beta)\right). \tag{78}$$

Had we used normal modes wewould have arrived at the expressions

$$\eta(t) = \sum_\nu \frac{e^{-iw_\nu t}}{1 + \sum_k \frac{c_k^2}{(w_\nu - \omega_k)^2}}, \quad \text{and} \tag{79}$$

$$\epsilon(t) = \mathrm{Re}\left(\frac{2i}{\eta(t)}\sum_{k,\nu}\frac{n(\beta)c_k^2}{w_\nu - \omega_k}\frac{e^{-iw_\nu t}}{1 + \sum_l \frac{c_l^2}{(w_\nu - \omega_l)^2}}\right). \tag{80}$$

We moreover would have noticed that, in order not to have inverted oscillators the following condition has to be fulfilled, $\omega > \sum_k c_k^2/\omega_k$. Since (77) can be hard to solve, methods to obtain approximate solutions are welcome. For instance, we can expand in powers of c_k to second order, $\alpha(t) = \exp(k_0 + ck_1 + c^2 k_2)$, to obtain

$$-i\Omega(t) - \Lambda(t) = -i\omega_0 t - \sum_k c_k^2 \int_0^t dt_1 \int_0^t dt_2 \mathrm{e}^{-i(\omega_k - \omega_0)t_2}. \tag{81}$$

We expect (81) to be valid for small c_k. It can be shown that this is the time dependent Born–Markov approximation, which is valid also for "strong" coupling and short times[16].

6 Experimental Characterization

We observe that the calculation of the mean energy, and of the entropy of the above examples of initial conditions only require the knowledge of $\Lambda(t)$. However, if for example one wants to measure the Wigner function of any field density matrix, one would need to know $\Omega(t)$ as well. To determine this function one takes advantage of the experimental setup to measure the Wigner function[4], in which a two level atom is prepared in the excited state $|e\rangle$, sent trough an array of two low–Q cavities R1 and R2 and one high–Q cavity C between them, and is detected eventually. The field in C is displaced by the operator $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$, by a microwave source connected to it. R1 and R2 behave as "rotation" operators in the Hilbert space of atomic states, $\mid e\rangle \to (\mid e\rangle + e^{i\xi} \mid g\rangle)/\sqrt{2}$, and $\mid g\rangle \to (-e^{-i\xi} \mid e\rangle + \mid g\rangle)/\sqrt{2}$, with $\xi = 0$ in R1 and $\xi = \pi/2$ in R2. The dispersive atom–field interaction in C produces entanglement: The field component associated with $\mid e\rangle$ suffers the action of the operator $\exp(i\pi(a^\dagger a + 1)/2)$, while the one associated with $\mid e\rangle$ suffers the action of $\exp(-i\pi a^\dagger a/2)$. It was shown in Ref. [4] that for this experimental arrangement

$$\Delta P = P_e - P_g = W(-\alpha, -\alpha^*, t)/2, \tag{82}$$

where P_e (P_g) is the probability to detect the probe atom in the upper (lower) state $|e\rangle$ ($|g\rangle$), and $W(-\alpha, -\alpha^*, t)$ is the value of the Wigner function corresponding to the field in thehigh–Q cavity at the time t the probe atom exits this cavity. Notice that, for the sake of convenience the normalization of W has been changed by a factor of π. The Wigner function of the coherent state (43) is

$$W(\alpha, \alpha^*, t) = 2e^{-2(\sigma(t)-\alpha)(\sigma(t)-\alpha)^*}. \tag{83}$$

Let's notice that $W(\alpha, \alpha^*, t)$ presents a maximum at $\alpha = \sigma(t) = \sigma_0 \exp(-\Lambda(t) - i\Omega(t))$. Since Λ can be determined from a photocounting experiment, we just need to measure $\Omega(t)$. We choose $\alpha = \alpha(t) = \sigma_0 \exp(-\Lambda(t)) \exp(-i\Phi(t))$, and adjust $\Phi(t)$ to maximize ΔP. If $\Phi(t) = \Omega(t) \mathrm{mod}(2\pi)$, then we obtain $\Delta P = 1$, otherwise, it will be smaller than one, $\Delta P = \exp(-8\sigma_0^2 \exp(-2\Lambda) \sin^2((\Omega - \Phi)/2))$. Choosing different exit times t we can map $\Omega(t)$. Notice that what we do, in fact, is find the right $\alpha(t)$ that brings the field at the cavity C to the vacuum. If we succeed in so doing the probe atom is rotated in R1, getting in a superposition of upper and lower states, flies free till R2 where the rotation is reversed, and goes to the detectors again in the upper state.

7 Concluding Remarks

In conclusion, we have shown that, if the optical cavities can be modelled through the hamiltonian (1), then only *three, experimentally measurable* real functions are necessary to characterize their behavior: the mean photon number of an initial ground state, the instantaneous frequency and the rate of change of the number of excitations. Measuring thes quantities once will enable one to get the Wigner function for any initial condition. So, provided the adequacy of the RWA model has been tested the present work provides for an alternative way to construct the time evolution of Wigner's functions.

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